Problem 1. Page 349, Chapter 8, Section 3, Problem 5: Find the general solution of the differential equation

\[ y' \cos x + y = \cos^2 x. \]

**Answer**
Divide everything by \( \cos x \) to get

\[ y' + \frac{1}{\cos x} y = \cos x. \]

Set \( y = uv \), and differentiate. You get

\[ u(v' + \frac{1}{\cos x} v) + vu' = \cos x. \]

Pick \( v \) such that \( v' + \frac{1}{\cos x} v = 0 \); this yields

\[ v = \exp(-\int \frac{dx}{\cos x}) \]

and so

\[ u' = \frac{\cos x}{v} = \cos x \exp\left(\int \frac{dx}{\cos x}\right) \]

from which you conclude

\[ u = A + \int dx \cos x \exp\left(\int \frac{dx}{\cos x}\right) \]

and to get \( y \) multiply this back by \( v \); you find

\[ y = uv = A + \int dx \cos x \exp\left(\int \frac{dx}{\cos x}\right) \exp\left(\int \frac{dx}{\cos x}\right). \]

Complete the integrals yourself.
**Problem 2.** Page 349, Chapter 8, Section 3, Problem 12: Find the general solution of the differential equation
\[
\frac{dx}{dy} = \cos y - x \tan y.
\]

**Answer**

Treat \( x \) as the unknown function and \( y \) as the known variable. Set \( x = uv \) and differentiate - now prime stands for a derivative with respect to \( y \). You get
\[
u'(v + v \tan y) + vu' = \cos y
\]
and so you choose \( v = \exp\left(\int \tan y \, dy\right) = \cos y \), so that when you substitute this into the remaining equation for \( u \), you get
\[
u' = \frac{\cos y}{v} = \frac{\cos y}{\cos y} = 1
\]
whence
\[
u = A + y
\]
and so
\[x = uv = (A + y) \cos y.\]
**Problem 3.** Page 352, Chapter 8, Section 4, Problem 10: Solve

\[(y^2 - xy)dx + (x^2 + xy)dy = 0.\]

**Answer**

Rewrite the equation as

\[y' = \frac{y^2 - xy}{x^2 + xy}.\]

Define new variable \(z = \frac{y}{x}\) (See page 351 on Homogeneous Equations). Plug it back into the differential equation, and differentiate using the chain rule. You find from \(y = xz\) that

\[xz' + z = \frac{x^2z^2 - x^2z}{x^2 + x^2z},\]

and so you can cancel the \(x^2\) on the right hand side, which leaves it to be a function of \(z\) alone. You have

\[xz' = -z - \frac{z^2 - z}{1 + z},\]

which you may rewrite as

\[xz' = -2\frac{z^2}{1 + z},\]

and realize that this equation separates variables. It then reduces to

\[\frac{1 + z}{z^2}dz = -2\frac{dx}{x},\]

and you can integrate it to get

\[\int \frac{1 + z}{z^2}dz = -2\ln x\]

or doing the left hand side integral explicitly,

\[\ln z - \frac{1}{z} = -2\ln x\]

or in terms of \(y\),

\[\ln y - \frac{x}{y} = -\ln x.\]
Problem 4. Page 359, Chapter 8, Section 5, Problem 12: Solve

\[(2D^2 + D - 1)y = 0.\]

Answer

Factorize the $D$-polynomial as

\[(2D^2 + D - 1) = (D + 1)(2D - 1).\]

With this, you know that the solutions of

\[(D + 1)(2D - 1)y = 0.\]

are exponentials $\exp(-x)$ and $\exp(x/2)$. Thus the general solution is

\[y = A\exp(-x) + B\exp(x/2).\]
Problem 5. Page 359, Chapter 8, Section 5, Problem 21: By the method used in solving (5.4) to get (5.9) show that the solution of the third-order equation

\[(D - a)(D - b)(D - c)y = 0\]

is

\[y = c_1e^{ax} + c_2e^{bx} + c_3e^{cx}\]

if \(a, b, c\) are all different, and find the solutions if two or three of the roots of the auxiliary equation are equal. DO NOT WORRY ABOUT GENERALIZING TO HIGHER-ORDER EQS - IGNORE THIS LAST PART.

Answer

Use the factorization argument: Note that any solution of

\[(D - a)y = 0\]

or

\[(D - b)y = 0\]

or

\[(D - c)y = 0\]

is also a solution of

\[(D - a)(D - b)(D - c)y = 0.\]

When all the roots \(a, b, c\) are different, then the solutions of the three first order equations, \(\exp(ax)\), \(\exp(bx)\) and \(\exp(cx)\) respectively, are linearly independent (check that the Wronskian is different from zero!). Thus the general solution of the third order equation is a linear combination of the three:

\[y = c_1e^{ax} + c_2e^{bx} + c_3e^{cx}.\]

If two of the roots, say \(a\) and \(b\) coincide, then the solutions of two first order equations are linearly dependent (ie they are identical), but then you have to solve separately the equation

\[(D - a)^2y = 0\]

by two integrations. That yields

\[y = (Ax + B)\exp(ax)\]

and then the solution of the third order equation is a linear combination of this and \(\exp(cx)\), since they are linearly independent; so

\[y = (Ax + B)\exp(ax) + C\exp(cx).\]

When all three roots are equal, a direct integration of

\[(D - a)^3y = 0\]

will yield

\[y = (Ax^2 + Bx + C)\exp(ax).\]