Homework 4 Solutions

7.2.4. (a) Set $A_0 = A(0), B_0 = B(0)$. Separating variables and using a partial fraction expansion obtain

$$\alpha \int dt = \alpha t = \int \frac{dC}{(A_0 - C)(B_0 - C)} = \frac{1}{B_0 - A_0} \int \left( \frac{1}{A_0 - C} - \frac{1}{B_0 - C} \right) dC.$$ 

Thus $\ln \frac{A_0 - C}{B_0 - C} = (A_0 - B_0)\alpha t + \ln \frac{A_0}{B_0}$.

Rewrite this as $C(t) = \frac{A_0 B_0 [e^{(A_0 - B_0)\alpha t} - 1]}{A_0 e^{(A_0 - B_0)\alpha t} - B_0}$. Then $C(0) = 0$.

(b) From $\int \frac{dC}{(A_0 - C)^2} = \alpha t$ get $\frac{1}{A_0 - C} = \alpha t + \frac{1}{A_0}$,

which yields $C(t) = \frac{\alpha A_0^2 t}{1 + \alpha A_0 t}$. Again $C(0) = 0$.

7.2.7. If $\frac{\partial \varphi}{\partial x} = P(x, y)$ then $\varphi(x, y) = \int_{x_0}^{x} P(X, y) dX + \alpha(y)$ follows.

Differentiating this and using $\frac{\partial \varphi}{\partial y} = Q(x, y)$ we obtain

$$Q(x, y) = \frac{d\alpha}{dy} + \int_{x_0}^{x} \frac{\partial P(X, y)}{\partial y} dX,$$

so

$$\frac{d\alpha}{dy} = Q(x, y) - \int_{x_0}^{x} \frac{\partial Q(X, y)}{\partial X} dX.$$
So \( \frac{d\alpha}{dy} = Q(x_0, y) \) and \( \alpha(y) = \int_{y_0}^{y} Q(x_0, Y) \, dY \). Thus

\[
\varphi(x, y) = \int_{x_0}^{x} P(X, y) \, dX + \int_{y_0}^{y} Q(x_0, Y) \, dY.
\]

From this we get \( \frac{\partial \varphi}{\partial x} = P(x, y) \) and

\[
\frac{\partial \varphi}{\partial y} = \int_{x_0}^{x} \frac{\partial P(X, y)}{\partial y} \, dX + Q(x_0, y) = \int_{x_0}^{x} \frac{\partial Q(X, y)}{\partial X} \, dX + Q(x_0, y) = Q(x, y).
\]

7.2.12. Separating variables we get

\[
-\frac{bt}{m} = \ln(g - \frac{b}{m}v) - \ln(A \frac{b}{m})
\]

with \( A \) an integration constant. Exponentiating this we obtain

\[
v(t) = \frac{mg}{b} - Ae^{-bt/m}, \quad \text{thus} \quad v_0 = v(0) = \frac{mg}{b} - A.
\]

Hence \( v(t) = \left( v_0 - \frac{mg}{b} \right) e^{-bt/m} + \frac{mg}{b} \). Set \( v_0 = 0 \) here.

The velocity dependent resistance force opposes the gravitational acceleration implying the relative minus sign.

7.3.3. Try solution \( e^{mx} \). The condition on \( m \) is \( m^3 - 3m + 2 = 0 \), with roots \( m = 1, m = 1, m = -2 \). Two independent solutions for \( m = 1 \) are \( e^x \) and \( xe^x \), so the general solution to the ODE is \( c_1 e^x + c_2 xe^x + c_3 e^{-2x} \).

7.4.1. For \( P(x) = -\frac{\zeta}{1-x^2} \), \( Q(x) = \frac{li(t+1)}{1-x^2} \),

\( (1+x)P \) and \( (1+x)^2Q \) are regular at \( x = \pm 1 \), respectively. So these are regular singularities.

As \( z \to 0 \), \( 2z - \frac{2/z}{1-1/z^2} = 2(z + \frac{z}{1-z^2}) \) is regular, and \( \frac{Q(z^{-1})}{z^4} = \frac{l(l+1)}{z^2(z^2-1)} \sim z^{-2} \) diverges. So \( \infty \) is a regular singularity.
7.5.3. If $a_1 k(k + 1) = 0$ with $a_1 \neq 0$, then $k = 0$ or $k = -1$.

(a) $k = 0$ sets $a_1 k(k + 1) = 0$ where $a_1$ remains undetermined.

(b) If $k = 1$ then the indicial equation $a_1 k(k + 1) = 0$ requires $a_1 = 0$. 